17-4 Continuous Distributions

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For continuous random variables, we have to handle things a little differently because:

We cannot build a probability distribution by listing all of the possible outcomes and assigning each a probability. There are an infinite amount of possible outcomes.

If we try to assign probabilities to each possible outcome, the probabilities will no longer sum to 1. There are an infinite amount of possible outcomes.

Probability Density Function (PDF)

Let x be a continuous random variable. The probability density function, f(x), of the random variable is a function with the following properties.

1.) f(x) > 0 for all x

2.) The area under the probability density function over all values of the random variable = 1.

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

3.) The probability that x equals any point x = a is 0.

4.) The probability that $a \le x \le b$

$$P(a \le x \le b) = \int_a^b f(x)dx$$

$$P(a < x < b) = P(a \le x \le b) = P(a < x \le b) = P(a \le x < b)$$

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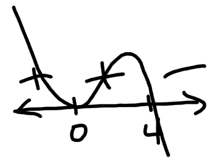
Ex1. Consider the function defined by

$$f(x)=\begin{cases} \frac{3}{64}\left(4x^2-x^3\right) & if \quad 0\le x\le 4 \\ 0 & otherwise \end{cases}$$



a.) Show that f is a well defined probability density function

(1.) $f(x)\ge 0$ for all x ✓

(2.) $\int_0^4 f(x)\,dx=1$

(1.) $\frac{3}{64}(4x^2-x^3)\ge 0$ on $]0,4[$?

$\frac{3}{64}x^2(4-x)=0$

$x^2=0 \qquad 4-x=0$

$x=0 \qquad x=4$

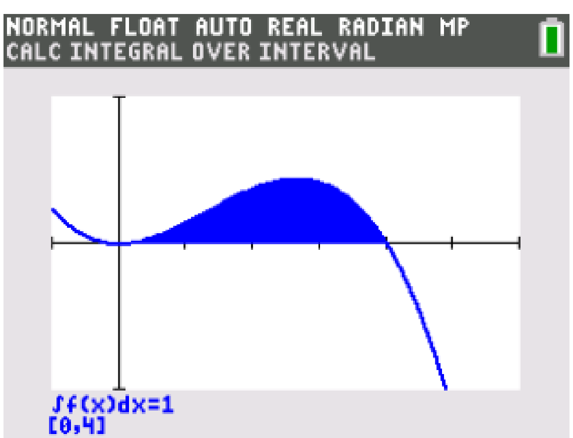
(2.) $\int_0^4 \frac{3}{64}(4x^2-x^3)\,dx \stackrel{?}{=} 1$

$=\frac{3}{64}\int_0^4 (4x^2-x^3)\,dx$

$=\frac{3}{64}\left[\frac{4}{3}x^3-\frac{1}{4}x^4\right]\Big|_0^4$

$=\frac{3}{64}\left[\left(\frac{4}{3}(64)\right)-\frac{1}{4}(256)\right)-0\right]$

$= 4-3=\boxed{1}$ ✓

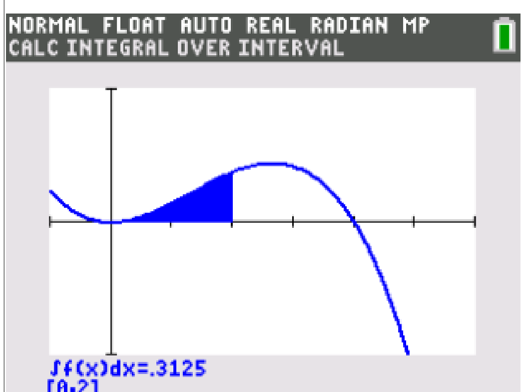


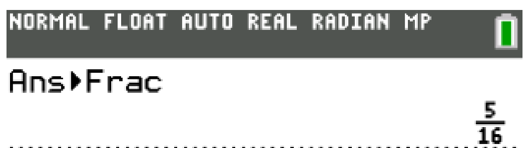
$$f(x)=\begin{cases} \frac{3}{64}\left(4x^2-x^3\right) & \text{if} \quad 0\le x\le 4 \\ 0 & \text{otherwise}\end{cases}$$

b.) Find the probability that $0 < x < 2$.

$$\frac{3}{64}\int_0^2 4x^2 - x^3 \, dx$$

$$\frac{3}{64}\left[\frac{4}{3}x^3 - \frac{x^4}{4}\right]_0^2 \quad \frac{3}{64}\left[\frac{32}{3} - 4\right] = \frac{5}{16}$$



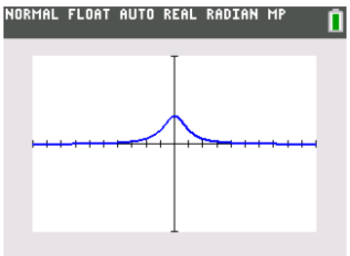


Ex2. A continuous random variable is defined as

$$f(x) = \frac{1}{\pi(x^2+1)} \quad \text{for} \quad -\infty \leq x \leq \infty$$

a.) Show that f is a well defined probability density function

$f(x) > 0 \; \forall \; x$ ✓



(1.) x^2 is + so x^2+1 is + so $\pi(x^2+1)$ is +. If the denominator is always + then $\frac{1}{\pi(x^2+1)}$ is alway +.

(2.) $\int_{-\infty}^{\infty} \frac{1}{\pi(x^2+1)}dx \overset{?}{=} 1$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx = \frac{1}{\pi}\left[\int_{-\infty}^{0} \frac{1}{x^2+1}dx + \int_{0}^{\infty} \frac{1}{x^2+1}dx\right]$$

$$= \frac{1}{\pi}\left[\lim_{a\to-\infty}\int_a^0 \frac{1}{x^2+1} + \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}dx\right]$$

$$= \frac{1}{\pi}\left[\lim_{a\to-\infty} \tan^{-1}x\Big|_a^0 + \lim_{b\to\infty} \tan^{-1}x\Big|_0^b\right]$$

$$= \frac{1}{\pi}\left[\lim_{a\to-\infty}(\tan^{-1}0 - \tan^{-1}a) + \lim_{b\to\infty}(\tan^{-1}b - \tan^{-1}0)\right]$$

$$= \frac{1}{\pi}\left[-\lim_{a\to-\infty}\tan^{-1}a + \lim_{b\to\infty}\tan^{-1}b\right]$$

$$= \frac{1}{\pi}\left[+\left(+\frac{\pi}{2}\right) + \frac{\pi}{2}\right]$$

$$= \frac{1}{\pi}(\pi) = \boxed{1} \checkmark$$

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$$f(x)=\frac{1}{\pi(x^2+1)} \quad \text{for} \quad -\infty \le x \le \infty$$

b.) Find the probability that x < 1

$$\int_{-\infty}^{1} \frac{1}{\pi(x^2+1)}\,dx = \frac{1}{\pi}\int_{-\infty}^{1} \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{\pi} \lim_{a\to-\infty} \int_{a}^{1} \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{\pi} \lim_{a\to-\infty} \tan^{-1} x \Big|_a^1$$

$$= \frac{1}{\pi}\left[\lim_{a\to-\infty} \tan^{-1} 1 - \tan^{-1} a\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4} - \lim_{a\to-\infty} \tan^{-1} a\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4} + \frac{\pi}{2}\right] = \frac{1}{\pi}\left[\frac{3\pi}{4}\right] = \boxed{\frac{3}{4}}$$

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Ex3. A continuous random variable x has pdf defined by

$$f(x)=\begin{cases}\dfrac{k}{x^2+x} & if \quad 1\le x\le 2\\ 0 & \text{otherwise}\end{cases}$$

Find the value of k so that f(x) represents a properly defined pdf

$\int_1^2 \frac{k}{x^2+x}\,dx = 1$ $\quad \int_1^2 \frac{k}{x(x+1)}$ $\quad \frac{k}{x(x+1)} = \frac{A}{x} + \frac{B}{(x+1)}$

$k = (x+1)A + Bx$

$x=0$: $k = 1A + 0B$

$k = A$

$x=-1$: $k = 0A - 1B$

$-k = B$

$\int_1^2 \frac{k}{x}\,dx + \int_1^2 \frac{-k}{(x+1)}\,dx$

$k\ln|x|\Big|_1^2 + -k\ln|x+1|\Big|_1^2$

$k\ln|2| - \cancel{k\ln|1|} - k\ln|3| + k\ln|2| = 1$

$k(2\ln|2| - \cancel{\ln|1|} - \ln|3|) = 1$

$k(2\ln|2| - \ln|3|) = 1$

$k(\ln|\frac{2^2}{3}|) = 1$ $\quad K = \frac{1}{\ln|4/3|}$

$\int_1^2 \frac{(1/\ln|4/3|)}{x^2+x}\,dx = 1$

K is positive, and because the integral is between 2 positive numbers, $f(x) > 0$

Cumulative Distribution Functions

A cumulative distribution function, F(x), of a random variable x with a density function p(t) is defined by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} p(t)dt$$

F(x) gives us the proportion of the population having values smaller than x.

Any cumulative distribution function has the following properties:

1.) F(x) is non-decreasing

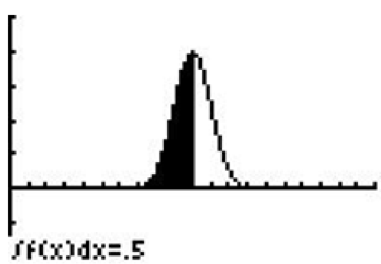
2.) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

Mean, Median, and Mode of a Continuous Distribution

Mean: $\mu = E(x) = \int_{-\infty}^{\infty} x \cdot p(x)dx$

Mode: this is the max value of f(x)

Median: The median for a random variable x that has a pdf of p(x) is a value m such that 50% of the values of x are less or equal m.



$$\int_{-\infty}^{m} p(x)dx = \frac{1}{2}$$

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Ex4. Given the pdf

$$f(x) = \begin{cases} \frac{3}{64}\left(4x^2 - x^3\right) & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a.) Find the mean, median, and mode.

$E(x) =$ Mean: $\int_0^4 \frac{3}{64} x(4x^2 - x^3)\,dx$

$= \int_0^4 \frac{3}{16}x^3 - \frac{3}{64}x^4\,dx = \frac{3}{64}x^4 - \frac{3}{320}x^5 \Big|_0^4$

$= \left[\frac{3}{64}(4)^4 - \frac{3}{320}(4)^5\right] - \left[\frac{3}{64}(0)^4 - \frac{3}{320}(0)^5\right]$

$= \left[\frac{3}{64}\cdot 256 - \frac{3}{320}\cdot 1024\right] - [0] = 12 - 9.6$

$x = 2.4$ is the Mean

$m =$ Median:

$\int_0^m \frac{3}{64}(4x^2 - x^3)\,dx = \frac{1}{2}$ $\quad \int_0^m \frac{3}{16}x^2 - \frac{3}{64}x^3\,dx = \frac{1}{2}$

$= \frac{1}{16}x^3 - \frac{3}{256}x^4 \Big|_0^m = \frac{1}{2}$

$= \frac{1}{16}m^3 - \frac{3}{256}m^4 = \frac{1}{2}$ solve graphically

$x \approx 2.45709$ is the median

mode: $f = \frac{3}{64}(4x^2 - x^3) = \frac{3}{16}x^2 - \frac{3}{64}x^3$

$f' = \frac{3}{8}x - \frac{9}{64}x^2 = 0$

$x\left(\frac{3}{8} - \frac{9}{64}x\right) = 0$

$x = 0$ $\quad \frac{3}{8} - \frac{9}{64}x = 0$

$\frac{3}{8} = \frac{9}{64}x$

$x = 8/3$

candidates: $x = 0, 8/3, 4$

$f(0) = \frac{3}{64}(4\cdot 0^2 - 0^3) = 0$

$f(8/3) = \frac{3}{64}\left(4\cdot \frac{8}{3}^2 - \frac{8}{3}^3\right) = 3/64\left(4\cdot\frac{64}{9} - \frac{512}{27}\right)$

$= 3/64\left(\frac{256}{9} - \frac{512}{27}\right) = 3/64\left(\frac{768}{27} - \frac{512}{27}\right)$

$= 3/64\left(\frac{256}{27}\right) = \frac{4}{9}$

$f(4) = 3/64(4\cdot 4^2 - 4^3)$

$= 3/64(64 - 64) = 0$

$f'' = 3/8 - 9/32x$

$f''(8/3) = 3/8 - 9/32(8/3)$

$= 3/8 - 3/4 = -3/8$ con dwn max

$x = 8/3$ is the mode

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$$f(x) = \begin{cases} \frac{3}{64}\left(4x^2 - x^3\right) & if \quad 0 \le x \le 4 \\ 0 & otherwise \end{cases}$$

b.) Is f(x) skewed right/left?

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$$f(x)=\begin{cases} \frac{3}{64}\left(4x^2-x^3\right) & if \quad 0\le x\le 4 \\ 0 & otherwise \end{cases}$$

c.) Find the first and third quartile and construct a box and whisker plot.

1st quartile

$$\int_0^f \frac{3}{64}(4x^2-x^3)\,dx=\frac{1}{4}$$

$$\int_0^f \frac{3}{16}x^2-\frac{3}{64}x^3\,dx=\frac{1}{2}$$

$$=\frac{1}{16}x^3-\frac{3}{256}x^4\Big|_0^f=\frac{1}{4}$$

$$=\frac{1}{16}f^3-\frac{3}{256}f^4=1/4$$

solve graphically

$x\approx 1.82529$

3rd quartile

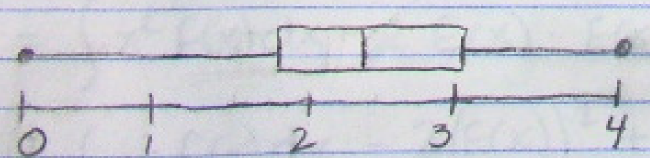
$$\int_0^t \frac{3}{64}(4x^2-x^3)\,dx=\frac{3}{4}$$

$$=\frac{1}{16}x^3-\frac{1}{256}x^4\Big|_0^t=3/4$$

$$=\frac{1}{16}t^3-\frac{1}{256}t^4=3/4$$

Solve graphically

$x\approx 3.02791$



Variance and Standard Deviation

Discrete: $\text{var}(x) = \sum (x-\mu)^2 \cdot p(x)dx$

Continuous: $\text{var}(x) = \int_{-\infty}^{\infty} (x - E(x))^2 \cdot f(x)dx$

$$\text{var}(x) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx - \left(E(x)\right)^2$$

Standard Deviation: $\sigma = \sqrt{\text{var}(x)}$

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$$f(x)=\begin{cases} \frac{3}{64}\left(4x^2-x^3\right) & if \quad 0\le x\le 4 \\ 0 & otherwise \end{cases}$$

d.) Find the variance and Standard Deviation.

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d) find the standard deviation & variance

$var(x) = \int_0^4 x^2 \cdot \frac{3}{64}(4x^2-x^3)\,dx - (2.4)^2$

$\frac{3}{64}\int_0^4 4x^4 - x^5\,dx - 5.76$

$\frac{3}{64}\left[\frac{4}{5}x^5 - \frac{1}{6}x^6\right]_0^4 - 5.76 = \frac{3}{64}\left[\frac{4}{5}(4)^5 - \frac{1}{6}(4)^6\right] - 5.76$

$Var(x) = 6.4 - 5.76 = \boxed{0.64}$

S.d: $\sigma = \sqrt{.64} = \boxed{.8}$

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